CERTAIN ASPECTS OF LINEAR GROUPS OVER FIELDS

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OUTLINE AND AIM OF THE PROJECT

The main aim of the project is to

- Study linear groups over fields, in the process we have also studied
  - Representations.
  - Homogeneous Components.
  - Schur's Lemma and Applications.
  - Soluble and Nilpotent Groups.
- Study Completely Reducible Linear Groups.
- Study System of Imprimitivity.
- Study examples of Linear Groups.
- Study Soluble Linear Groups.

MAIN RESULTS

- Maschke-Schur Theorem.
- Clifford's Theorem.
- Blichfelt's Theorem.
- Burnside's Theorem.
- Suprunenko's Theorem.
- Zassenhaus Theorem.

INTRODUCTION:

- Let F be a field and V a vector space over F of finite dimension n.
- F_n is the algebra of all n x n matrices over field F.
- End_F(V) is the algebra of all F-endomorphisms of V.
- GL(n,F) is the group of units of F_n.
- GL(V,F) is the group of units of End_F(V).

- Let $\mathfrak{B} = \{v_1,v_2,......,v_n\}$ be a fixed basis of V over F, then $End_F(\mathrm{V}) \cong F_n$, as algebras over F, via the isomorphism

  $[\ ]_{\mathfrak{B}}$:$End_F(\mathrm{V}) \longrightarrow F_n$, defined as,

  $$\mathrm{T} \longrightarrow [\ \mathrm{T}\ ]_{\mathfrak{B}}.$$

- GL(V,F) $\cong$ GL(n,F).

  Subgroups of GL(V,F), GL(n,F) are called **Linear Groups**

- Let R be a ring. A subgroup of GL(n,R), the group of all $n \times n$ invertible matrices over the ring R, is called R-Linear Group. We shall study only the linear groups over fields.

Representations and Completely Reducible Modules

• Let G be a group. A representation ρ of G over a field F with representation space V is a group homomorphism
$\rho : G \longrightarrow$ GL(V,F) where V is a vector space over field F.

• Every representation ρ of group G over field F determines a group action

$$. : G \times V \longrightarrow V$$
$$(g, v) \longrightarrow g.v = \rho(g)(v)$$

• There is one to one correspondence between representations of group G over field F and the left FG-modules.

Let X$\subseteq End_F$(V).Then F{X} will denote the F-subalgebra generated by X.

•X is called irreducible if V as an F{X}-module is irreducible.

• Let V be an FG-module and W be a subspace of V, W is called G-invariant subspace of V if gw $\in$ W for all g in G and w in W.

•A representation ρ is called irreducible if the only G-invariant subspaces of the associated left FG-module V are (0) and V, otherwise it is called reducible.

•A representation ρ is irreducible if and only if V is irreducible where V is the associated left FG-module determined by ρ.

•A subgroup G of GL(V,F) is irreducible if V as FG-module is irreducible, where V is made FG-module via the action T.v $= T$(v) of G on V.

• A module M over a ring R is called completely reducible if it is direct sum of irreducible R-modules and a representation ρ of group G over field F is called completely reducible if it arises from completely reducible FG-module via the module operation determined by ρ.

Proposition: If G is a subgroup of GL(n,F) then there exists positive integers $r, n_1, n_2, ..., n_r$ for each 1 $\leq$ i $\leq$ r, an irreducible representation ρ_i of G in GL(n_i,F), and an element x of GL(n,F)such that
$n_1 + n_2 + ... + n_r =$ n and

$$x^{-1}gx = \begin{pmatrix} \rho_1(g) & & * \\ & .. & \\ 0 & & .. \\ & & \rho_r(g) \end{pmatrix} \text{ for all g in G.}$$

HOMOGENEOUS COMPONENTS

Let R be ring and let M be an R-module, let $\mathfrak{S}_M$ be a complete family of all mutually non isomorphic irreducible submodules of M.

If M is a completely reducible R-module, then M $= \oplus_{i\in I} S_i$, S_i is irreducible R-module for each i $\in$ I.

For each S $\in \mathfrak{S}_M$, let I_S=$\{$i $\in I \mid S_i \cong S\}$ and

$$M_{I_S} = \oplus_{i\in I_S} S_i$$
$$= \oplus_{i\in I} \{S_i \mid S_i \cong S\}$$

M_{I_S} is called homogeneous component of type S and it follows that M is direct sum of its homogeneous components, i.e.
M $=\oplus_{S\in \mathfrak{S}_M} M_{I_S}$.

Theorem: Let M be a completely reducible R-module and let N be an irreducible submodule of M_{I_S}, S $\in \mathfrak{S}_M$, the homogeneous component of type S, then $N \cong S$.

Theorem: Let M be a completely reducible R-module and if N is irreducible submodule of M then N is contained in some homogeneous component of M. In particular

$$M_{I_S} = \sum_{\substack{N irreducible \\ N \cong S}} N.$$

Theorem: Let M be a completely reducible R-module and then every homogeneous component of M is invariant under all R-endomorphisms of M.

SCHUR's LEMMA

Schur's Lemma: Let R be a ring and U and V be irreducible R-modules then any R-module homomorphism of U to V is either an isomorphism or the zero map. End_R(V)is a division ring.

Corollary: Let F be a field and V be an irreducible FG-module then End_{FG}(V)is a division algebra over F with F in its center.

Theorem: Let F be an algebraically closed field and G an irreducible subgroup of GL(n,F), then

i) $C_{F_n}(\mathrm{G}) = \mathrm{F}\ 1_n$.

ii) $Z(\mathrm{G}) = \mathrm{G} \bigcap F^*\ 1_n$.

Corollary: Let F be an algebraically closed field,G be an abelian irreducible subgroup of GL(n,F) then n = 1.

•If F is an algebraically closed field and if V is an irreducible FG-module of finite F-dimension then every element of End_{FG}(V)determines a scaler linear transformation of V.

• If H is subgroup of a group G, then a left transversal of H to G (or a complete set of left coset representatives)is a subset T of G consisting of precisely one element from each distinct left coset of H in G.

Main theorem on completely reducible linear groups

Theorem: Let G be a group and H be a subgroup of finite index m in G , let F be a field of characteristic 0 or characteristic relatively prime to m and V an FG -module which is completely reducible as FH-module. Then V is completely reducible as FG-module.

Corollary: If G is a linear group over field F such that char F $=0$ or char F $=$ p >0 and prime to m, H be a subgroup of G of finite index m in G and if H is completely reducible then G is completely reducible.

Theorem: (**Maschke-Schur**) Let G be a locally finite subgroup of GL(n,F) and either char F =0 or char F = p > 0 and G contains no element of order p, then G is a completely reducible linear group.

Corollary: If G is finite subgroup of GL(n,F) and either char F = 0 or char F = p > 0 where p does not divide o(G), then G is completely reducible.

System of imprimitivity and Linear Groups

System of Imprimitivity

Let G be a group and V an FG-module.If V contains r non trivial subspaces $V_1, V_2, ..., V_r$ such that $V = V_1 \oplus V_2 \oplus .... \oplus V_r$ and for each i, $1 \leq i \leq r$ and each g $\in$G there is k=k(i,g), $1 \leq k \leq r$ such that $gV_i = V_k$, then $\{V_1, V_2, ..., V_r\ \}$ is called **system of imprimitivity** of G in V.

If {V } is the only system of imprimitivity of G in V then we say G acts **primitively** on V otherwise G acts imprimitively on V.

- If G is a subgroup of GL (n,F) and V is a vector space of dimension n over F, then G is called primitive or imprimitive according to whether G acts primitively or imprimitively on V, via the natural action of G on V.

- If G is completely reducible and primitive, then G is irreducible.

•Let G be a group and let V be an FG-module and let $\{V_1, V_2, ...$ $......., V_r\}$ be a system of imprimitivity of G in V, then G acts transitively on $\{V_1, V_2, ..., V_r\}$ if for each pair i and j there exists a g in G such that gV_i =V_j.

Theorem: Let G be a group and let V be an FG-module and let $\{V_1, V_2, ..., V_r\}$ be a system of imprimitivity of G in V, then G has a normal subgroup of finite index dividing n!, where $\dim_F$V =n.

Theorem: If $\{V_1, V_2, ..., V_r\}$ is a system of imprimitivity of G in V,where V is an irreducible FG-module. Then G acts transitively on $\{V_1, V_2, ..., V_r\}$.

Theorem: Let $\{V_1, V_2, ..., V_r\}$ be a system of imprimitivity of group G in V, if V is an irreducible FG-module, then each V_i is irreducible as FS_i-module, where $S_i = \{$g $\in$ G: g$V_i = V_i\}$.

CLIFFORD's THEOREM AND LINEAR GROUPS

Theorem:(A.H.Clifford) Let G be a group and let H be a normal subgroup of G and V an irreducible FG-module of finite F-dimension, then

1. V is completely reducible into irreducible FH-modules of equal F-dimensions.

2. If $W_1, W_2, ..., W_r$ are homogenous components of V as FH-module then $\{W_1, W_2, ..., W_r\}$ is system of imprimitivity of G in V.

Proof: Let U be an irreducible FH-submodule of V, which exists since
dim_FV is finite. Let g$\in$ G, then gU is a subspace of V. Clearly
ϕ :U $\longrightarrow$ gU
u $\longrightarrow$ ϕ(u)=gu

is a one-one, onto, linear map. Hence U $\cong_F$ gU and dim_F U $=$ dim_F gU

Let h $\in$ H,

$g^{-1}hg \in$ H, since H is normal in G.
U is an FH-module, so $g^{-1}hg$U $\subseteq$ U, hence

hgU $=gg^{-1}hg$U
$\subseteq$ gU, for all h in H, so gU is an FH-submodule of V.

We now show that gU is irreducible FH-submodule of V. Let U_1 be a proper non-zero FH-submodule of gU, as proved above $g^{-1}U_1$ is proper FH-submodule of g^{-1}gU = U this contradicts that U is irreducible as FH-module so gU has no proper FH-submodules and hence gU is irreducible FH-submodule of V.

W = $\sum_{g\in G}$ gU, is an FG-submodule of V and since U is irreducible W $\neq$ (0). V is irreducible FG-module, hence W = V, so V = $\sum_{g\in G}$ gU, hence V is completely reducible into irreducible FH-modules of equal F-dimension.

Now, let $\{W_1, W_2, ..., W_r\}$ be homogenous components of V as FH-module.

Let U_1, U_2 be irreducible FH-submodules of W_i, a homogenous component of V as FH-module, hence $U_1 \cong_{FH} U_2$. Let $\phi : U_1 \longrightarrow U_2$ be the FH isomorphism.

Let g $\in$ G then since U_1, U_2 are irreducible FH-submodules, as proved above gU_1 and gU_2 are irreducible FH-submodules,

$\phi^g = g\phi g^{-1}$:$gU_1 \longrightarrow gU_2$,

ϕ^g (gu + gw) = $g\phi g^{-1}$ (g(u +w))

= gϕ(u +w)

= gϕ(u) +gϕ(w)

$= g\phi g^{-1}$ (gu) + $g\phi g^{-1}$ (gw)

$= \phi^g$ (gu) + ϕ^g (gw) for all gu in gU_1 and all gw in U_1

also,
ϕ^g (agu)= a ϕ^g (gu) for all a in F.

For h $\in$H, ϕ^g (hgu) = $g\phi g^{-1}$ (hgu)

$= g\phi(g^{-1}$hgu)

$g^{-1}hg \in H$ since H normal in G and ϕ is an FH- isomorphism so,

$g\phi$ ($g^{-1}hg$u)= $gg^{-1}hg\phi$ (u)

$= gg^{-1}hg\phi$ (u)

$= hg\phi g^{-1}$ (gu)

$= h\phi^g$ (gu)

$\Rightarrow$ ϕ^g (hgu) $= h\phi^g$ (gu) for all h in H and gu in gU_1.

Hence ϕ^g is an FH-isomorphism so $gU_1 \cong_{FH} gU_2$ and both irreducible FH-modules, so $\exists$ j =j(g,i)
s.t. $gU_1, gU_2 \subseteq W_j$

so, $gU_1 + gU_2 \subseteq W_j$

so, $gW_i \subseteq W_j$, U_1, U_2 any irreducible FH-submodules of W_i rest follows from Theorem for homogenous component W_i.

$\{W_1, W_2, ..., W_r\}$ are homogenous components of V as FH-module, so

V $= W_1 \oplus W_2 \oplus .... \oplus W_r$

$= gW_1 \oplus gW_2 \oplus .... \oplus gW_r$

If $gW_i \subsetneq W_j$ then since V is has finite F-dimension,
dim$_F$V $= dim_F(gW_1) + dim_F(gW_2) + ... + dim_F(gW_r)$
$< dim_F W_1 + dim_F W_2 + ... + dim_F W_r$
=dim$_F$V, is not possible, hence $gW_i = W_j$.

For each g in G and 1 $\leq$ i $\leq$ r there is j=j(i,g) such that $gW_i = W_j$, hence $\{W_1, W_2, ..., W_r\}$ a system of imprimitivity of G in V. The proof is complete.

• A subnormal series of a group G is a finite sequence of subgroups (1)=$G_0 \trianglelefteq G_1 \trianglelefteq$ .... $\trianglelefteq G_n$=G. A subgroup H of G is called subnormal if it is a term of a subnormal series of group G.

Corollary: Let H be a subnormal subgroup of a completely reducible linear group, then H is completely reducible.

Proof: Let G be a completely reducible linear group over field F and H be a subnormal subgroup of G, so there is a subnormal series (1)=$\trianglelefteq H = H_1 \trianglelefteq$ .... $\trianglelefteq H_{k-1} \trianglelefteq H_k$=G. Let V be a vector space over F of dimension n where G$\leq$ GL(n,F), making V a FG-module via the natural action of G on V, then V is completely reducible as an FG-module. Hence,
V= $V_1 \oplus V_2 \oplus$ .... $\oplus V_r$, each V_i is irreducible FG-module.

$H_{k-1} \trianglelefteq G$, by Clifford's Theorem, each V_i is completely reducible as
FH_{k-1}-module.
Hence V is completely reducible as FH_{k-1}-module and hence continuing like this V is completely reducible as
(FH_1=)FH-module. So H is a completely reducible linear group.

Theorem(Blichfeldt) If F is algebraically closed field and G an irreducible subgroup of GL(n,F), if A is abelian normal subgroup of G, then there is a is system of imprimitivity $\{V_1, V_2, ..., V_r\}$ of group G in V, such that $C_G(A) = \bigcap_{i=1}^{r} N_G(V_i)$, where $N_G(V_i) = \{\text{ g} \in G : gV_i = V_i \}$, further if G is primitive as well, then every abelian normal subgroup of G lies in center of F_n.

Theorem(Burnside) Let A be an irreducible F-subalgebra of F_n. If C_{F_n} (A) $= F1_n$, then A $= F_n$.

Corollary(Burnside) If F is algebraically closed and G is an irreducible subgroup of GL(n,F) then G contains n^2 linearly independent elements.

Soluble and Nilpotent Groups

Soluble Groups

Let, $\delta^{(0)}(G) = G$

$\delta^{(1)}(G) = [\ \delta^{(0)}(G)\ ,\ \delta^{(0)}(G)\ ]$

$\qquad = [\ G\ ,G\ ] = G'$, the commutator subgroup of G.

$\vdots$

$\delta^{(n+1)}(G) =[\ \delta^{(n)}(G)\ ,\ \delta^{(n)}(G)\ ]$

$G = \delta^{(0)}(G) \supseteq \delta^{(1)}(G) \supseteq \delta^{(2)}(G)......................\supseteq \delta^{(n)}(G) \supseteq \delta^{(n+1)}(G)...........$ is the derived series of G.

A group G is called soluble if $\delta^{(n)}(G) = (1)$, for some n. If $\delta^{(n)}(G) = (1)$ and $\delta^{(n-1)}(G) \neq (1)$, then G is called soluble group of length n.

Nilpotent Groups

Let, $\gamma_1(\mathrm{G}) = \mathrm{G}$

$\gamma_2(\mathrm{G}) = [\ \gamma_1(\mathrm{G})\ ,\ \mathrm{G}\ ]$

$\vdots$

$\gamma_{n+1}(\mathrm{G}) =[\ \gamma_n(\mathrm{G})\ ,\ \mathrm{G}\ ]$

$\mathrm{G} = \gamma_1(\mathrm{G}) \supseteq \gamma_2(\mathrm{G}) \supseteq \gamma_3(\mathrm{G}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \supseteq \gamma_n(\mathrm{G}) \supseteq \gamma_{n+1}(\mathrm{G}) \ldots\ldots\ldots\ldots$
is the Lower Central series of G.

Let $\mathrm{Z}_0(G) = (1)$

$\frac{Z_{n+1}(G)}{Z_n(G)} = Z(\frac{G}{Z_n(G)})$

$(1) = Z_0(\mathrm{G}) \subseteq Z_1(\mathrm{G}) \subseteq Z_2(\mathrm{G}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \subseteq Z_n(\mathrm{G}) \subseteq Z_{n+1}(\mathrm{G}) \ldots\ldots\ldots\ldots$ is the Upper Central series of G.

A group G is called nilpotent if $\gamma_c(\mathrm{G}) = (1)$, for some c. If $\gamma_{c+1}(\mathrm{G}) = (1)$ and $\gamma_c(\mathrm{G}) \neq (1)$, then G is called nilpotent group of class c.

Theorem: A group G is soluble if and only if it has a finite subnormal series with abelian factors, i.e., there exists (1)=$G_0 \trianglelefteq G_1$.............. $\trianglelefteq G_n = G$ in which factor groups $\frac{G_{i+1}}{G_i}$ are abelian.

Theorem: If G is a soluble group and (1) $\neq N \trianglelefteq$ G, then N contains a abelian normal subgroup of G.

Theorem: If G is a finite soluble group, then every minimal normal subgroup of G is elementary abelian.
Further If V is a minimal normal subgroup of a finite soluble group G, then G acts on V as a group of linear transformations.

EXAMPLES OF LINEAR GROUPS

GENERAL AND SPECIAL LINEAR GROUPS

Let F be a field and V a vector space over F of dimension n.

General linear group is the group, GL(n,F) $\cong$ GL(V ,F).

Special linear group

SL(n,F) ={ A $\in$ GL(n,F) | det A =1 }

SL(V,F) ={ T $\in$ GL(V,F) | det T =1 }, SL(n,F) $\cong$ SL(V,F).

Theorem: If F is a finite field having q elements, then GL(V,F),SL(V,F) are finite groups and

$$\mid GL(V,F) \mid = \mid GL(n,F) \mid = (q^n - 1)(q^n - q).......(q^n - q^{n-1})$$

$$\mid SL(V,F) \mid = \mid SL(n,F) \mid = \frac{(q^n-1)(q^n-q).......(q^n-q^{n-1})}{q-1}$$

Theorem:
i)GL(n,F) is semidirect product of SL(n,F) by F^*.
ii)SL(n,F) is generated by elementary transvections.

TRIANGULAR, UNITRIANGULAR AND DIAGONAL GROUPS .

Let F be a field, then

1. *Triagular subgroup* of GL(n,F) denoted by *Tr(n,F)* is the group of all upper triangular matrices in GL(n,F).

2. *Unitriagular subgroup* of GL(n,F) denoted by *U(n,F)* is the group of all upper triangular matrices with diagonal entries as 1.

3. *Diagonal subgroup* of GL(n,F) denoted by *D(n,F)* is the group of all diagonal matrices in GL(n,F).

Theorem: The commutator subgroup of Tr(n,F) is U(n,F) unless, when characteristic of field F is two.

Theorem: The unitriangular group, U(n,F), $n > 1$ is nilpotent of class n-1. It follows that there exists nilpotent groups of arbitrary class.

Theorem: U(n,F) and Tr(n,F) are soluble groups.

Theorem: If char F $= 0$ then U(n,F) is torsion free, and if char F $= p > 0$ then U(n,F) has finite exponent.

Theorem: Let F be a field and G be a finite group of order n, then G can be imbedded in GL(n,F).

Soluble Linear Groups

Lemma: Let H and L be subgroups of GL(n,F) such that [H,L] $\subseteq$ F$^*1_n$. Then K = C$_H(L)$ is normal in H, $\mid H : K \mid \leq n^2$ and the elements of any transversal of K to H are linearly independent over F.

Theorem: Let F be algebraically closed and A an irreducible subgroup of GL(n,F) which is nilpotent of class 2. Then $\mid A : Z(A) \mid = n^2$.

Theorem: (**Suprunenko**)Let F be an algebraically closed field, n be an integer greater than one, G a soluble, primitive irreducible subgroup of GL(n,F) and H a normal subgroup of G. Then G contains normal subgroups

(1) $\subseteq$ Z $\subseteq$ A $\subseteq$ H $\subseteq$ G, such that

i) Z = H $\cap$ Z(G).

ii) $\frac{A}{Z}$ is abelian and $\mid \frac{A}{Z} \mid = r^2$, for some integer r dividing n.

iii) $\mid \frac{H}{A} \mid \leq r^2\,!$.

Proof: F is algebraically closed and G is primitive irreducible subgroup of GL(n,F), also Z(G) is abelian normal subgroup of GL(n,F), hence by Blichfeldt's Theorem Z(G) lies in center of F_n. If K is any other abelian normal subgroup of G then K $\subseteq F^* 1_n$ and hence K $\subseteq$ Z(G).
Let Z = H $\cap$ Z(G), then since H and Z(G) are normal in G, so Z is normal in G. Let A be subgroup of G containing Z, maximal subject to restrictions,

A $\subseteq$ H, A $\trianglelefteq$ G, $\frac{A}{Z}$ is abelian, its existence is guaranteed by Zorn's Lemma.

We shall prove, this Theorem in three parts.

1.C$_H(A)$ = Z.

z $\in$ Z = H $\cap$ Z(G)

$\Rightarrow$ z $\in$ Z(G) and z $\in$ H

$\Rightarrow$ za = az, for all a in A.

$\Rightarrow$ z $\in$ $C_H(A)$

$\Rightarrow$ Z $\subseteq$ $C_H(A)$. ....(1)

Claim : $C_H(A) \trianglelefteq G$

Let g $\in$ G, x $\in$ $C_H(A)$
$\Rightarrow$ x $\in$ H and xa = ax for all a in A.

$g^{-1}xg\ \in H$, since H $\trianglelefteq$ G.

Also, $(g^{-1}xg)$a $= g^{-1}xg$a

$= g^{-1}xgag^{-1}g$

$= g^{-1}xa_1g$, where $a_1\ = gag^{-1}$ $\in$ A, A$\trianglelefteq$ G.

$= g^{-1}a_1xg$, since x $\in$ C$_H(A)$

$= g^{-1}gag^{-1}xg$

$= a(g^{-1}xg)$

$\Rightarrow\ g^{-1}xg \in$ C$_H(A)$, for all g in G and x in C$_H(A)$.

$\Rightarrow$ $C_H(A) \trianglelefteq G$.

Also, $A \trianglelefteq G$
so, $A \cap C_H(A) \trianglelefteq G$.
We note that $A \cap C_H(A)$ is abelian as well.

So $A \cap C_H(A)$ is abelian normal subgroup of G and hence as discussed before, $A \cap C_H(A) \subseteq Z(G)$
Hence, $A \cap C_H(A) \subseteq Z(G) \cap H = Z$

Also by (1), $Z \subseteq C_H(A)$, hence $Z \subseteq C_H(A) \cap A$, since Z is contained in A.

From above two arguments, it follows that $Z = C_H(A) \cap A$.
.............(2)

G is soluble, so its subgroup $C_H(A)$ is soluble and also $\frac{G}{Z}$ is soluble.

By (1), Z $\subseteq$ $C_H(A)$.

Since $C_H(A) \trianglelefteq$ G, $\frac{C_H(A)}{Z} \trianglelefteq \frac{G}{Z}$

We show that Z can not be properly contained in C_H(A).

If Z $\subsetneq$ $C_H(A)$, then $\frac{C_H(A)}{Z} \neq$ Z

Hence by Theorem 4.7, $\frac{C_H(A)}{Z}$ contains a nontrivial abelian normal subgroup of $\frac{G}{Z}$, say $\frac{K}{Z}$.

so, Z $\subsetneq$ K $\subseteq$ $C_H(A)$ and $\frac{K}{Z}$ is abelian..........(3)

Claim: $\frac{KA}{Z}$ is abelian normal subgroup of $\frac{G}{Z}$

Let Zka $\in \frac{KA}{Z}$, k $\in$ K, a $\in$ A, then for any g in G

$ZgZkaZg^{-1} = Zgkag^{-1}$

$= Zgkg^{-1}gag^{-1}$

$= Zk_1a_1$, $k_1 = gkg^{-1} \in$ K, $\;a_1 = gag^{-1} \in$ A since K and A are normal in G.

So, $ZgZkaZg^{-1} \in \frac{KA}{Z}$

Hence, $\frac{KA}{Z}$ is normal subgroup of $\frac{G}{Z}$.

Let h,k $\in$ K, a,x $\in$ A, then

ZkaZhx = Zkahx

=Zkhax, since h $\in$ K $\subseteq$ C_H(A)

= ZkZhZaZx

= ZhZkZxZa, since $\frac{A}{Z}$, $\frac{K}{Z}$ are abelian.

= Zhkxa

= Zhxka, since k $\in$ K $\subseteq$ C_H(A)

= ZhxZka

Hence $\frac{KA}{Z}$ is abelian.

$K \subseteq C_H(A)$, so $KA \subseteq C_H(A) \subseteq H$, also $KA \trianglelefteq G$
so, $A \subseteq KA \subseteq H$, and $\frac{KA}{Z}$ is abelian normal subgroup of $\frac{G}{Z}$

By maximality of A, $KA \subseteq A$, hence $K \subseteq A$, Also $K \subseteq C_H(A)$.

So, $Z \subsetneq K \subseteq C_H(A) \cap A = Z$, a contradiction.

Hence $C_H(A) = Z$.

2.$\mid A : Z \mid = r^2$

We first show that A is nilpotent of class 2.

$Z(A) \subseteq C_H(A) = Z$ by 1.
Also, if $z \in Z = Z(G) \cap H \subseteq Z(G)$ and $a \in A$

$\Rightarrow$ za = az
$\Rightarrow$ z $\in$ Z(A)
$\Rightarrow$ Z $\subseteq$ Z(A)

Hence, Z(A) = Z

We now consider the upper central series of A.

$Z_1(A) = Z(A)$

$\frac{Z_2(A)}{Z_1(A)} = Z(\frac{A}{Z_1(A)}) = Z(\frac{A}{Z})$

$\frac{A}{Z}$ is abelian, so $Z(\frac{A}{Z}) = \frac{A}{Z}$

Hence $Z_2(A) = A$.

Z(A) $\neq$ A, otherwise if A $=$ Z(A) $=$ Z, $\frac{A}{Z}$ is trivial and $\mid A : Z \mid =$ 1, so we can assume that Z(A) $\neq$ A.

Hence, (1) $=Z_0(A) \subseteq Z_1(A) \subseteq Z_2(A) = A$ and A is nilpotent of class 2.

Let V be FG-module of F-dimension n, via the natural action of G on V. Then by Clifford's Theorem, A is completely reducible. So V as FA-module is completely reducible. If $V_1, V_2, ..., V_m$ are homogenous components of V as FA-module, each of dimension r, V $= V_1 \oplus V_1 \oplus .... \oplus V_m$, n=mr. Each V_i is irreducible FA-module of dimension r. So by Corollary 4.16 $\mid A : Z(A) \mid = r^2$.

3. C$_H(\frac{A}{Z})$ $=$ A.

Let c $\in$ C$_H(\frac{A}{Z})$,

cZaZ = aZcZ, for all a in A, also, Z normal in G so aZ = Za.

$\Rightarrow$ [a,c] $\subseteq$ Z = Z(A), for all a in A..............(6)

Let $\phi_c : \frac{A}{Z} \longrightarrow Z$

Za $\longrightarrow$ [a,c]

let x,y $\in$ A

$y^{-1}x^{-1}c^{-1}x = y^{-1}x^{-1}c^{-1}xcc^{-1}$

$= y^{-1}[x,c]c^{-1}$

$= [x,c]y^{-1}c^{-1}$, follows from (6).

$\Rightarrow y^{-1}x^{-1}c^{-1}x = [x,c]y^{-1}c^{-1}$

$\Rightarrow y^{-1}x^{-1}c^{-1}xyc = [x,c]y^{-1}c^{-1}yc$

$\Rightarrow [xy,c] = [x,c][y,c]$, for all x,y in A

So, ϕ_c(Zxy) = ϕ_c(Zx) ϕ_c(Zy), for all Zx , Zy in $\frac{A}{Z}$

$\Rightarrow \phi_c \ \in Hom(\frac{A}{Z}, Z)$

Define, ψ : $C_H(\frac{A}{Z}) \longrightarrow Hom(\frac{A}{Z}, Z)$

$$c \longrightarrow \phi_c$$

Then ψ is a homomorphism and x $\in$ Ker $\psi \Leftrightarrow \phi_x$ is identity map, i.e.

id : $\frac{A}{Z} \longrightarrow$ Z

Za $\longrightarrow$ 1

$\Leftrightarrow$ [a,x] $=$ 1, for all a in A.

$\Leftrightarrow$ Ker $\psi =$ Z.

Hence $\frac{C_H(\frac{A}{Z})}{Z} \cong Im\psi$, $\frac{A}{Z}$ finite abelian and Z $\subseteq$ F$^*1_n$

so $\mid \frac{C_H(\frac{A}{Z})}{Z} \mid \leq \mid Hom(\frac{A}{Z}, Z) \mid \leq \mid \frac{A}{Z} \mid$
Also, A $\subseteq C_H(\frac{A}{Z})$
So, $C_H(\frac{A}{Z}) =$ A.
It follows that $\frac{H}{A}$ is isomorphic to a subgroup of $Aut(\frac{A}{Z})$.

Hence $\mid \frac{H}{A} \mid \leq r^2!$, since $\frac{A}{Z}$ finite, and using 2.

This completes the proof.

Corollary:(Zassenhaus) Let F be algebraically closed field and G be a soluble primitive irreducible subgroup of GL(n,F). Then center of G consists of scaler matrices and has finite index in G.

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